Dynamische Systemen

Final Exam

20/04/2005

Before starting ANY of the 4 excercises, PLEASE read all of them, CARREFULLY. Then CHOOSE 2 excercises amoung the 4 proposed. Additionally you may do more than expected, and it will be a plus. Whatch out however with consistency.

1 A map on the plane

Consider the following map

$$H_{\gamma_0,\gamma_1,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (x,y) \mapsto (\gamma_0 + \gamma_1 - x^2 - by, x).$$

1-1 Assume $b < 0$. Show that $H_{\gamma_0,\gamma_1,b}$ is not the time 1 of a smooth vector field on $\mathbb{R}^2$.

1-2 Assume now that $\gamma_0 = 0$, $|\gamma_1| << 1$ and $0 < b << 1$. Consider the following set

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid \forall n \in \mathbb{N}\ H^n(x,y) \in [-1,1] \times [-1,1]\}.$$

Describe the topology of Ω.

1-3 Show that $H_{\gamma_0,\gamma_1,1}$ has no sinks.

1-4 Recall that a set $\Lambda \subset \mathbb{R}^2$ is an attractor if Λ is compact and if there exists an neighbourhood $\mathcal{B}$ of Λ such that

$$\lim_{n\to\infty} d\Big((H^n_{\gamma_0,\gamma_1,b}(q), \Lambda\Big) = 0\ \forall q \in \mathcal{B},$$

where d is the usual metric on $\mathbb{R}^2$.

HINT: Assume that $\mathcal{B}$ is compact. Show that there exists an integer n such that

$$d\Big((H^n_{\gamma_0,\gamma_1,b}(\mathcal{B}), \mathcal{B}\Big) > 0,\ \forall q \in \mathcal{B},$$

and get a contradiction.

2 Hamiltonian system

Consider the following system

$$\begin{cases} \dot{x} &= ax - mxy, \\ \dot{y} &= bxy - \delta y, \end{cases} \tag{1}$$

where $x > 0$, $y > 0$ a, b, δ and m are positive parameters.

2-1 Show that for any initial value, the trajectory of the flow is a periodic orbit.

HINT: Look at the title of this section! look at (or go to) question [2-3]!

2-2 Deduce from [2-1] and without calculus that for all parameter values, system (1) has a single singularity of center type.

2-3 Explain the title: "Hamiltonian system".

3 Planar phase portrait

Consider the map

$$G : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$$

- Find the singular point of G, i.e., the set of points where dG is not submersive and the singular values.
- Show that or $E \notin \{0, -\frac{1}{4}\}$ $G^{-1}(E)$ is either
  - empty,
  - a closed curve,
  - two disjoint closed curve,
- Study the case $E = -\frac{1}{4}$ and $E = 0$. Give a draw of $G^{-1}(0)$.

Consider the following system

$$\mathcal{X} : \begin{cases} \dot{x} &= -y - G(x, y) \cdot (x^3 - x) \\ \\ \dot{y} &= -x + x^3 - yG(x, y) \end{cases} \tag{2}$$

Denote by $\mathcal{X}_t$ the flow associated with $\mathcal{X}$. In what follows,

$$\Omega(p) = \cap_{t \in \mathbb{R}} \cap_{s > t} \mathcal{X}_t(p) \tag{3}$$

is the so-called Ω-limit set of q of the flow of $\mathcal{X}$

3-1 Find all the singularities of (2)

3-2 Determine their type, i.e., saddle/focus/center ...etc..

3-3 Show that $\mathcal{X}_t$ leaves $G^{-1}(0)$ invariant. What is $G^{-1}(0)\backslash\{0\}$?

3-4 Show that for all $-\frac{1}{4} < -b < 0 < a$, the flow $\mathcal{X}_t$ enters the set $G^{-1}\{-b < x < a\}$ and exits out of $G^{-1}\{-\frac{1}{4} < x < -b\}$.

3-5 Show that $\mathcal{X}$ does not possess any periodic orbit.

We admit the following property to hold: $\forall \mathcal{U}$ neighborhood of $G^{-1}(0)$, $\exists \varepsilon > 0$ such that $G^{-1}[-\varepsilon, \varepsilon] \subset \mathcal{U}$.

3-6 Take $p \notin G^{-1}\{0, -\frac{1}{4}\}$. Show that for all $\varepsilon > 0$ $\exists t \in \mathbb{R}$ such that $\mathcal{X}^t(p) \in G^{-1}[-\varepsilon, \varepsilon]$. Show that $\Omega(p)$ is either the union of a homoclinic orbit and a saddle point or the union of two homoclinic orbits and a saddle point.

4 Over Poincaré Bendixson theorem

4-1 Recall Poincaré Bendixson theorem for a vector field X on $\mathbb{R}^2$. For simplicity we shall assume that X is of Morse-Smale type and possesses a compact invariant region, i.e., there exists $\mathcal{B} \subset \mathbb{R}^2$ such that $\Phi_t(\mathcal{B}) \subset \mathcal{B}$ where Φ_t denotes the flow at the time t of X.

HINT: Look at the definition in (3).

4-2 Let X_γ, $\gamma \in \mathbb{R}$ be a family of planar vector field. Recall Poincaré Hopf index theorem on a smooth close curve Γ bounding a disk D.

Consider the following familly X_γ of vector field defined on $[0,\infty)\times[0,\infty)$

$$\begin{aligned} \dot{x} &= x\Big(1 - x - \frac{y}{\gamma x + \frac{1}{2}}\Big) \\ \dot{y} &= y\Big(-1 - y + \frac{x}{\gamma x + \frac{1}{2}}\Big). \end{aligned}$$

4-3 Show that X_γ possesses a compact invariant region as in excercise 4-1, of the form

$$\mathcal{B} = \{x \geq 0,\ y \geq 0,\ x + y \leq p\}$$

where $p > 0$.

HINT: Show that the function

$$f_p : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}, \ (x, y) \mapsto x + y - p$$

is such that $X(f_p) < 0$ for p suffciently large. Make a draw.

4-4 Determine the singularities on $\mathcal{B}$. Show that X possesses no more than 3 singularities in $\mathcal{Q} = \{x > 0, \ y > 0\}$. What bifurcation occurs at the point $C = (1, 0)$?

4-4 Show that there exists a open region set $\mathcal{N} \subset \mathbb{R}$ such that for all $\gamma \in \mathcal{N}$ X_γ has no singularities in $\mathcal{Q}$. Denote by Φ_t the flow at the time t of X. Show that

$$\forall q \in \mathcal{Q}, \ \lim_{t \longrightarrow infty} \Phi_t(q) = (1, 0).$$

4-5 Let $\beta > 1/2$. Consider a closed curve Γ bonding a disk D close to

$$\partial\mathcal{B} = \{x = 0, 0 \leq y \leq p\} \cup \{y = 0, 0 \leq x \leq p\} \cup \{x+y = p, x \geq 0, \ y \geq 0\}.$$

Show that the index of X on Γ is zero.

4-5 Let $\beta < 1/2$. Show that X possesses an attractor in $\mathcal{B}$.

HINT: Apply the Poincaré Bendixon theorem to unstable manifold $W^u(C)$.

4-6 From [4-5] deduce that in the case $\beta < 1/2$ and for Γ sufficiently close to the boundary $\partial\mathcal{B}$, the index of X on Γ is 1.

HINT: Assume X_γ to be of Morse-Smale type. Take $q \in W^u(C)$. Show that $\Omega(q)$ is aeither an attacting point or a limit cycle. Compute the index of $\Omega(q)$ in each case and comclude with Poincaré Hopf index theorem.